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TENTAMEN RELATIVISTIC QUANTUM MECHANICS

Monday 07-02-2011, 14.00-17.00

On the first sheet write your name, address and student number. Write your name on all other sheets.

This examination consists of four problems, with in total 18 parts. The 18 parts carry equal weight in determining the final result of this examination.

$\hbar = c = 1$. The standard representation of the 4×4 Dirac gamma-matrices is given by:

$$\gamma^0 = \begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & -\mathbb{1}_2 \end{pmatrix}, \qquad \gamma^k = \begin{pmatrix} 0 & \sigma_k \\ -\sigma_k & 0 \end{pmatrix}, \qquad \gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 0 & \mathbb{1}_2 \\ \mathbb{1}_2 & 0 \end{pmatrix}.$$

PROBLEM 1

A spinor field transforms under Lorentz transformations as

$$\psi'(x') = S(\Lambda)\psi(x)\,, \tag{1.1}$$

where Λ is the Lorentz transformation matrix and $x^{\mu\prime} = \Lambda^\mu{}_\nu x^\nu$.

1.1 Write down the Dirac equation for the free spinor field.

1.2 Prove that covariance of the Dirac equation under Lorentz transformations implies

$$S^{-1}(\Lambda)\gamma^\mu S(\Lambda) = \Lambda^\mu{}_\nu \gamma^\nu\,. \tag{1.2}$$

1.3 Consider a Lorentz transformation corresponding to a boost with velocity v in the x^1-direction, that is

$$x^{0\prime} = \gamma(x^0 - vx^1)\,, \quad x^{1\prime} = \gamma(-vx^0 + x^1)\,, \quad x^{2\prime} = x^2\,, \quad x^{3\prime} = x^3\,,$$

where

$$\gamma = \frac{1}{\sqrt{1-v^2}}\,.$$

Write the Lorentz matrix $\Lambda^\mu{}_\nu$ for this transformation. For infinitesimal Lorentz transformations we write

$$\Lambda^\mu{}_\nu = \delta^\mu_\nu + \epsilon^\mu{}_\nu\,. \tag{1.4}$$

Also give $\epsilon^\mu{}_\nu$ for the transformation of (1.3), in the approximation of infinitesimal v.

1.4 For infinitesimal Lorentz transformations we set

$$S = \mathbb{1} + \delta S\,.$$

Obtain the equation for δS using equations (1.2) and (1.4).

1.5 Obtain the solution for δS for the approximation of infinitesimal v.

PROBLEM 2

In the quantisation of the electromagnetic field one starts with the Lagrangian density

$$\mathcal{L}_1 = -\tfrac{1}{4} F_{\mu\nu} F^{\mu\nu} \,, \tag{2.1}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$.

2.1 The fields $A_\mu(x)$ are taken to be the canonical coordinates. Determine the corresponding canonical momenta. Why are these coordinates and momenta inconvenient for the standard procedure of canonical quantisation?

2.2 Argue that an arbitrary field $A_\mu(x)$ is physically equivalent to another field $A'_\mu(x)$ which satisfies

$$\partial_\mu A^{\mu\prime} = 0 \,.$$

2.3 Now we modify the Lagrangian density by including an additional term:

$$\mathcal{L}_2 = -\tfrac{1}{4} F_{\mu\nu} F^{\mu\nu} - \tfrac{1}{2}(\partial_\mu A^\mu)^2 \,. \tag{2.2}$$

Determine the canonical momenta for the Lagrangian density (2.2).

2.4 Derive the equation of motion for the field $A_\mu(x)$ from the Lagrangian density (2.2).

2.5 To make the quantum theory which follows from (2.2) equivalent to ordinary electrodynamics, which follows from (2.1), we have to somehow set $\partial_\mu A^\mu$ equal to zero. Explain, without going into extensive mathematical detail, how this should be done.

PROBLEM 3

Consider the theory of a scalar field $\phi(x)$, with the Lagrangian density

$$\mathcal{L} = \tfrac{1}{2}\partial_\mu \phi(x) \partial^\mu \phi(x) - \tfrac{1}{2} m^2 \phi(x)\phi(x).$$

The expansion of $\phi(x)$ in plane waves is

$$\phi(x) = \int \frac{d^3k}{\sqrt{(2\pi)^3 2\omega_k}} \left(a(k) e^{-ikx} + a^\dagger(k) e^{ikx}\right)_{k^0=\omega_k} .$$

3.1 What is the canonical momentum $\pi(x)$ corresponding to the coordinate $\phi(x)$?

3.2 Given that classically $\{\phi(t,\vec{x}), \pi(t,\vec{y})\}_{\text{PB}} = \delta^3(\vec{x}-\vec{y})$, what is the result of the equal-time commutation relation

$$[\phi(x), \pi(y)]_{x^0=y^0} \tag{1.2}$$

for the quantum operators ϕ and π?

3.3 Show that ($x^0 \neq y^0$!)

$$[\phi(x), \phi(y)] = \int \frac{d^3k}{(2\pi)^3 2\omega_k} \left(e^{-ik(x-y)} - e^{+ik(x-y)}\right)_{k^0=\omega_k} . \tag{1.3}$$

3.4 Using the result of (1.3) evaluate

$$[\phi(x), \partial_{y^0}\phi(y)]\,,$$

and show that in the limit $x^0 \to y^0$ the result of (1.2) is obtained.

PROBLEM 4

In a scattering process the particles in the initial and final state are considered to be free relativistic particles.

Consider the elastic scattering process between two electrons:

$$e_1 + e_2 \to e_3 + e_4,$$

with four-momenta k_1^μ, k_2^μ, k_3^μ, k_4^μ. The spatial momenta satisfy (center of mass frame)

$$\vec{k}_1 + \vec{k}_2 = 0\,.$$

The electrons have mass m.

4.1 What is the value of $(k_i)^\mu (k_i)_\mu$ for $i = 1, \ldots, 4$?

4.2 What is the value of $\vec{k}_3 + \vec{k}_4$?

4.3 Show that the energies of the four electrons are equal.

4.4 Choose a coordinate system such that the spatial momentum of e_1 is

$$\vec{k}_1 = (k, 0, 0)\,.$$

We now perform a Lorentz boost in the x^1 direction. On the momenta this acts as on the coordinates:

$$k_i^0{}' = \gamma(k_i^0 - v k_i^1)\,, \quad k_i^1{}' = \gamma(-v k_i^0 + k_i^1)\,, \quad k_i^2{}' = k_i^2\,, \quad k_i^3{}' = k_i^3\,,$$

where the index $i = 1, \ldots, 4$ indicates the four electrons and $\gamma = 1/\sqrt{1-v^2}$. Calculate $\vec{k}_3{}' + \vec{k}_4{}'$ in this new coordinate system.